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On the Ahlberg-Nilson Extension of Spline Interpolation: The g-Splines and Their Optimal Properties*

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INTRODUCTION

It is known that the Euler summation formula

$$\int_0^r f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + \frac{1}{2}f(r) + \frac{B_2}{2!}(f'(0) - f'(r)) + \frac{1}{2!}\int_0^r \bar{B}_2(x) f''(x)\,dx \tag{1}$$

is among all approximate quadrature formulas of this type which are exact for linear functions, the best in the sense of Sard. By this is meant the following: If

$$\int_0^r f(x)\,dx = \sum_0^r A_\nu f(\nu) + C_1 f'(0) + D_1 f'(r) + \frac{1}{2!}\int_0^r K(x) f''(x)\,dx \tag{2}$$

is an identity valid for all $f(x) \in C''$, the constants A_ν, C_1, D_1 and the kernel $K(x)$ being independent of $f(x)$, then

$$\int_0^r (K(x))^2\,dx > \int_0^r (\bar{B}_2(x))^2\,dx,$$

unless the formulas (1) and (2) are identical (see [1]).

It is also known that the best formula (1), of the type (2), *must* arise by integrating the corresponding spline interpolation formula, exact for linear functions, and interpolating the data

$$f(0), \ldots, f(r), f'(0), f'(r) \qquad \text{(see [2])}.$$

Evidently, this is the interpolation formula by natural cubic spline functions for the nodes $0, 1, \ldots, r$ of which 0 and r are *double* nodes.

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We likewise know that

$$\int_0^r f(x)\,dx = \frac{1}{2}f(0) + f(1) + \cdots + \frac{1}{2}f(r) + \frac{B_2}{2!}(f'(0) - f'(r))$$

$$+ \frac{B_4}{4!}(f'''(0) - f'''(r)) + \frac{1}{4!}\int_0^r \bar{B}_4(x) f^{(4)}(x)\,dx \qquad (3)$$

is among all approximate quadrature formulas of this type, which are exact for cubic polynomials, the best in the sense of Sard. Wishing to uphold the Newtonian pre-eminence of interpolation in problems of approximation of functionals, we raise the following question: *Does* (3) *also arise by integration from an appropriate interpolation formula?*

Here the term "appropriate" means that the interpolation formula should interpolate the data

$$f(0), \ldots, f(r), f'(0), f'(r), f'''(0), f'''(r) \qquad (4)$$

and be exact for cubic polynomials.

Evidently ordinary spline interpolation, with 0 and r as multiple nodes, will no longer do for the reason that the values of the second derivative $f''(0)$ and $f''(r)$ are not present among the data (4).

Numerous other examples of this kind (see [3, p. 83] and [4, p. 174]) point out the need for an extension of the theory of spline interpolation. Such an extension has recently been brilliantly achieved in the paper [4] by Ahlberg and Nilson. They generalize the "natural spline functions" to what they call "splines of interpolation." Because the natural term "generalized splines" already describes an extension in a different direction, we shall use here the term "g-splines."

We discuss here the g-splines for two reasons: firstly, in order to emphasize their importance and usefulness, secondly, in order to overcome some weak points in the theory as presented in [4]. A minor weakness in [4] is the inadequacy of the notations. More serious is the lack of conditions for the uniqueness of the solutions of the interpolation and best approximation problems discussed in [4]. The correct conditions are readily suggested if we consider our aims in their proper historical context. These aims may be described as follows:

Ordinary spline interpolation at simple nodes generalizes Lagrange's formula of polynomial interpolation. Likewise, spline interpolation with multiple nodes generalizes the Hermite interpolation problem. The purpose of g-splines is to generalize the interpolation problem first considered by G. D. Birkhoff in 1906 [5]. In this problem the data are again values of the function and of its derivatives but without Hermite's condition that only consecutive derivatives be used at each node. However, while in Lagrange and Hermite interpolation involving n data, there is always a unique polynomial solution of degree $n-1$, this is no longer the case in Birkhoff's problem. Here we may have no solutions of degree $n-1$ and if they exist, we may have infinitely many. A condition akin to Birkhoff's requirement that his problem be "normal" plays an equally important role in a discussion of Ahlberg's and Nilson's g-splines.

The following discussion is self-contained and requires no previous acquaintance with spline functions and their properties. Rather it contains an exposition of these properties in an essentially generalized form. These are derived along the lines first used by T. N. E. Greville [6] for the case of spline interpolation with simple nodes. The present writer has previously based such discussions on the so-called B-splines (see [7], where the B-splines are called fundamental spline functions). The B-splines remain indispensable in the theory of variation diminishing approximation methods (see [8], [9]). However, for interpolation problems Greville's approach is far shorter and more direct.

The paper consists of three parts which are sufficiently described by their headings. We might add that in the third part we state some open questions of interpolatory function theory concerning infinite expansions which are analogues of E. T. Whittaker's cardinal series.

I. The Interpolation Problem

1. The Hermite-Birkhoff Interpolation Problem.

Let

$$x_1 < x_2 < \cdots < x_k \tag{1.1}$$

be distinct reals. Let $e = \{(i, j)\}$ be a prescribed set of distinct ordered pairs (i, j) such that i assumes each of the values $1, 2, \ldots, k$, once or several times, while $j \in \{0, 1, \ldots, \ell\}$, the value $j = \ell$ being assumed for some pair (i, j). Furthermore, let $y_i^{(j)}$ be prescribed reals for each $(i, j) \in e$ and let us consider the problem of finding functions $f(x) \in C^\ell$ which satisfy the interpolatory conditions

$$f^{(j)}(x_i) = y_i^{(j)} \qquad \text{for} \qquad (i, j) \in e. \tag{1.2}$$

We may conveniently rephrase this description in terms of an "incidence matrix"

$$E = \| \epsilon_{ij} \| \qquad (i = 1, \ldots, k; j = 0, 1, \ldots, \ell) \tag{1.3}$$

having elements $\epsilon_{ij} = 0$ or 1. We think of ϵ_{ij} as being in the ith row and jth column. We require that each row of E, and also its last column, should contain some element $= 1$. The matrix E will likewise describe the set of equations (1.2) if we define the set e by

$$e = \{(i,j) \mid \epsilon_{ij} = 1\}. \tag{1.4}$$

Of importance is the integer

$$n = \sum_{i,j} \epsilon_{ij} \tag{1.5}$$

which tells us that the system (1.2) has n equations.

Interpolation problems (1.2) were first studied by G. D. Birkhoff[1] [5]. A noteworthy special case is obtained if we assume that E has the additional property:

If

$$0 \leqslant j' < j \qquad \text{and} \qquad \epsilon_{ij} = 1 \qquad \text{then} \qquad \epsilon_{i,j'} = 1. \tag{1.6}$$

It is then seen that at each node x_i the system (1.2) prescribes the value $f(x_i)$ and perhaps also a certain number of consecutive derivatives $f^{(j)}(x_i)$, for $j = 1,..., \alpha_i - 1$, say. Then (1.2) is what we may call an *Hermite interpolation problem* [1]. It is therefore appropriate to refer to (1.2) as an *Hermite-Birkhoff interpolation problem*, which we shall abbreviate to *HB*-problem.

DEFINITION 1. *Following Birkhoff we shall say that the HB-problem (1.2) is normal provided that (1.2) can always be solved uniquely by an $f(x) \in \pi_{n-1}$.*[2]

A *necessary* condition for (1.2) to be normal is the inequality

$$n > \ell \tag{1.7}$$

which we shall assume throughout our discussion. For if $n \leqslant \ell$ then $n - 1 < \ell$ and a $f(x) \in \pi_{n-1}$ could not possibly always satisfy the Eqs. (1.2) involving $f^{(\ell)}(x)$.

Clearly every Hermite system is normal.

Assuming (1.2) to be normal let us denote for each $(i,j) \in e$ by $L_{ij}(x)$ the unique element of π_{n-1} such that

$$L_{ij}^{(s)}(x_r) = \delta_{ir}\delta_{js} \qquad \text{if} \qquad (r, s) \in e,$$

or explicitly

$$L_{ij}^{(s)}(x_r) = \begin{matrix} 0 & \text{if} & (r,s) \neq (i,j) \\ 1 & \text{if} & (r,s) = (i,j). \end{matrix} \tag{1.8}$$

[1] Interpolation is a wonderful subject for name-dropping. Some of the greatest mathematicians have contributed to it: From Newton Lagrange, Gauss, Abel to Fejér, Birkhoff, Polya and numerous other greats. In truth, these mathematicians would not be quite so famous if they had done nothing else.

[2] We denote by π_k the class of real polynomials of degrees not exceeding k.

In terms of these fundamental functions we may express the unique solution of (1.2) in π_{n-1} by

$$f(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x)$$

Moreover, if $f(x)$ is an arbitrary function in C^ℓ we may write

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i) L_{ij}(x) + Rf, \tag{1.9}$$

where the right-hand sum represents the π_{n-1} which interpolates $f(x)$. The formula (1.9) may be referred to as the *Hermite-Birkhoff interpolation formula.* It is exact, i.e., $Rf = 0$, whenever $f(x) \in \pi_{n-1}$.

2. On Normal Systems and Related Concepts. The condition that the HB-problem (1.2) be normal may be equivalently expressed by the following requirement: *If*

$$P(x) \in \pi_{n-1}, \tag{2.1}$$

$$P^{(j)}(x_i) = 0 \qquad \text{if} \qquad (i,j) \in e, \tag{2.2}$$

then $P(x) \equiv 0$.

A closely related notion is described by

Definition 2. *Let m be a natural number. We say that the HB-problem (1.2) is m-poised provided that*

$$P(x) \in \pi_{m-1}. \tag{2.3}$$

$$P^{(j)}(x_i) = 0 \qquad \text{if} \qquad (i,j) \in e \tag{2.4}$$

imply that $P(x) = 0$.

Remarks. 1. *The HB-problem (1.2) is normal if and only if it is n-poised.*

2. *If (1.2) is m-poised then the inequality*

$$m \leqslant n \tag{2.5}$$

must hold. Indeed, if $m > n$ then $P(x) \in \pi_{m-1}$ depends on m parameters. But then the homogeneous system (2.4), having only n equations, must admit nontrivial solutions.

3. *If (1.2) is m-poised and $1 \leqslant m' < m$, then (1.2) is also m'-poised* Evidently so because $\pi_{m'-1} \subset \pi_{m-1}$.

4. *A non-normal system (1.2) may well be m-poised for some value $m < n$.* As an example let us consider the HB-problem

$$f(x_1) = y_1, \qquad f'(x_2) = y_2', \qquad f(x_3) = y_3, \qquad (x_2 = \tfrac{1}{2}(x_1 + x_3)). \tag{2.6}$$

Here

$$E = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{Vmatrix},$$

hence $\ell = 1$, $n = 3$.

Notice that (2.6) is *not* normal, i.e., not 3-poised, because

$$P(x) = (x - x_1)(x - x_3) \not\equiv 0$$

satisfies the conditions (2.1) and (2.2). However, notice that

$$(2.6) \textit{ is 2-poised.} \tag{2.7}$$

5. The condition that (1.2) be m-poised is easily expressed as follows: If

$$P(x) = \sum_0^{m-1} a_\nu \frac{x^\nu}{\nu!},$$

then the Eqs. (2.4) become

$$\sum_{\nu=0}^{m-1} a_\nu \frac{x_i^{\nu-j}}{(\nu - j)!} = 0 \qquad \text{for} \qquad (i,j) \in e,$$

where we write $x^r/r! = 1$ if $r = 0$, and $= 0$ if $r < 0$. Therefore, (1.2) *is m-poised if and only if the matrix*

$$\left\| \frac{x_i^{\nu-j}}{(\nu - j)!} \right\| \qquad \text{has rank} \qquad m, \tag{2.8}$$

where $\nu = 0,\ldots, m - 1$ *indicates the column, while to each* $(i,j) \in e$ *corresponds a row of the matrix.*

3. The g-Splines. For the remainder of this paper we assume that the natural number m satisfies the following condition:

$$\textit{The HB-problem (1.2) is m-poised} \qquad \text{(Definition 2).} \tag{3.1}$$

By (2.5) this assumption implies that $m \leqslant n$. For reasons which will appear later we shall even assume that

$$\ell < m \leqslant n. \tag{3.2}$$

Example. By (2.7) HB-problem (2.6) satisfies all these conditions with $\ell = 1$, $m = 2$ and $n = 3$.

We now enlarge the matrix E defined by (1.3) by adding to it $m - \ell - 1$ columns entirely composed of zero elements. In symbols let

$$E^* = \| \epsilon_{ij}^* \|, \qquad (i = 1,..., k; j = 0,..., m - 1) \tag{3.3}$$

where

$$\epsilon_{ij}^* = \begin{matrix} \epsilon_{ij} & \text{if} & j \leqslant \ell, \\ 0 & \text{if} & j = \ell + 1,..., m - 1. \end{matrix} \tag{3.4}$$

Observe that $E^* = E$ if $m = \ell + 1$.

DEFINITION 3. *A function $S(x)$ is called a natural g-spline for the knots $x_1 ,..., x_k$, the matrix E^*, and order m, provided that it satisfies the following conditions:*

I. $S(x) \in \pi_{2m-1}$ *in* (x_i , x_{i+1}), $(i = 1,..., k - 1)$,

II. $S(x) \in \pi_{m-1}$ *in* $(-\infty, x_1)$ *and in* $(x_k , +\infty)$,

III. $S(x) \in C^{m-1}(-\infty, \infty)$,

IV. *If* $\epsilon_{ij}^* = 0$ *then* $S^{(2m-j-1)}(x)$ *is continuous at* $x = x_i$, *i.e.,*

$$S^{(2m-j-1)}(x_i - 0) = S^{(2m-j-1)}(x_i + 0).$$

We denote by

$$\mathscr{S}_m = \mathscr{S}_m(E^*; x_1 ,..., x_k) \tag{3.5}$$

the totallity of natural g-splines just described.

That $\mathscr{S}_m$ is not void is shown by the inclusion relation

$$\pi_{m-1} \subset \mathscr{S}_m . \tag{3.6}$$

Indeed, if $S(x) \in \pi_{m-1}$ then $S(x)$ satisfies all conditions from I to IV.

EXAMPLES AND SPECIAL CASES. 1. *The Lagrange problem.* We assume that $\ell = 0$, hence $n = k$ and

$$E = \| 1, 1,..., 1 \|^T \tag{3.7}$$

the corresponding problem (1.2) being

$$f(x_1) = y_1 ,..., f(x_n) = y_n . \tag{3.8}$$

Let us inspect the corresponding class $\mathscr{S}_m$ of g-splines, where by (3.2) m is such that $0 < m \leqslant n$. By (3.7) and (3.4) we see that

$$\epsilon_{ij}^* = 0 \qquad (j = 1,..., m - 1)$$

and this for each $i = 1,\ldots, n = k$. By condition IV we conclude that

$$S^{(2m-j-1)}(x) \quad \text{is continuous at} \quad x = x_i \quad \text{for} \quad j = 1,\ldots, m-1,$$

or equivalently, that

$$S^{(\nu)}(x) \quad \text{is continuous at} \quad x = x_i \quad \text{for} \quad \nu = m, m+1,\ldots, 2m-2.$$

This conclusion and condition III show that

$$S(x) \in C^{2m-2}(-\infty, \infty).$$

We conclude: *The class $\mathscr{S}_m$ of g-splines corresponding to the Lagrange interpolation problem (3.8) is identical with the class of natural spline functions of degree* $2m-1$ *having the simple knots* $x_1,\ldots, x_n$, [see 6].

2. THE HERMITE PROBLEM. Let us now assume that (1.2) is an Hermite problem

$$f(x_i) = y_i, \quad f'(x_i) = y_i',\ldots, \quad f^{(\alpha_i-1)}(x_i) = y_i^{(\alpha_i-1)} \quad (i = 1,\ldots, k). \tag{3.9}$$

Here $\ell = \max \alpha_i - 1$ and we can choose any m such that $\max_i \alpha_i \leqslant m \leqslant n$. Since

$$\epsilon_{ji}^* = 0 \quad \text{if} \quad j = \alpha_i, \alpha_i + 1,\ldots, m-1,$$

we conclude from condition IV that

$$S^{(2m-j-1)}(x) \quad \text{is continuous at} \quad x = x_i \quad \text{if} \quad j = \alpha_i,\ldots, m-1,$$

or equivalently, that

$$S^{(\nu)}(x) \quad \text{is continuous at} \quad x = x_i \quad \text{if} \quad \nu = m, m+1,\ldots, 2m-\alpha_i-1.$$

Combining this with condition III we conclude that

$$S(x) \in C^{2m-\alpha_i-1} \quad \text{near} \quad x = x_i \quad (i = 1,\ldots, k). \tag{3.10}$$

We recognize in conditions I, II, and (3.10) the characteristic properties of the natural spline functions of degree $2m-1$ *having* x_i $(i = 1,\ldots, k)$ *as a multiple knot of multiplicity* α_i $(\alpha_i \leqslant m)$.

4. INTERPOLATION BY g-SPLINES. From the theory of spline interpolation we know that the natural spline functions of degree $2m-1$ corresponding to the problems (3.8) and (3.9), respectively, always solve their respective problems uniquely. In the present section we establish the similar more general result for g-splines.

THEOREM 1. *Let the Hermite-Birkhoff problem (1.2) be m-poised, while m satisfies the inequalities (3.2). Then (1.2) with prescribed $y_i^{(j)}$ has a unique solution*

$$S(x) \in \mathscr{S}_m(E^*; x_1, \ldots, x_k). \tag{4.1}$$

PROOF. (1.2) is evidently a linear problem. Setting up all polynomial components of $S(x)$ with indeterminate coefficients we obtain a total of

$$m + (k-1)\,2m + m = 2mk \text{ unknowns.}$$

We now count the number of equations: From condition III (Definition 3) we obtain mk equations. The number of equations resulting from condition IV is equal to the number of vanishing elements of E^*. Since their total number is mk, while $n = \sum \epsilon_{ij}$ is the number of nonvanishing elements, we obtain from IV $mk - n$ equations. Finally (1.2) furnishes n further equations and we obtain a total of $mk + (mk - n) + n = 2mk$ equations.

The number of equations being equal to the number of unknowns, it suffices to establish the following:

If

$$S(x) \in \mathscr{S}_m \qquad \text{and} \qquad S^{(j)}(x_i) = 0 \qquad \text{for} \qquad (i,j) \in e \tag{4.2}$$

then

$$S(x) = 0 \qquad \text{for all all real } x. \tag{4.3}$$

We select a finite interval $I = [x_0, x_{k+1}]$ such that $x_0 < x_1$, $x_k < x_{k+1}$ and start by showing that (4.2) implies that

$$J \equiv \int_I (S^{(m)}(x))^2\, dx = 0. \tag{4.4}$$

First we write

$$J = \int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \cdots + \int_{x_k}^{x_{k+1}} (S^{(m)}(x))^2\, dx$$

and now we integrate by parts repeatedly each of these integrals according to the scheme

$$\int_{x_i}^{x_{i+1}} S^{(m)} S^{(m)}\, dx = S^{(m)} S^{(m-1)} \Big|_{x_i}^{x_{i+1}} - S^{(m+1)} S^{(m-2)} \Big|_{x_i}^{x_{i+1}} + \cdots$$

$$\pm S^{(2m-1)} S \Big|_{x_i}^{x_{i+1}} \mp \int_{x_i}^{x_{i+1}} S^{(2m)} S\, dx.$$

Now observe the following: 1. Each of the very last integrals on the right is $= 0$ because $S(x) \in \pi_{2m-1}$ in each interval by conditions I and II.

2. From the "finite parts" we obtain at each x_i ($i = 1,..., k$) a sum of terms

$$\sum_{j=0}^{m-1} \pm \Delta_i^{(j)},$$

where

$$\Delta_i^{(j)} = \underset{at\ x=x_i}{\text{jump of}}\ S^{(2m-j-1)}(x)\, S^{(j)}(x), \qquad (j = 0,..., m-1).$$

Since $S(x) \in C^{m-1}$ by condition III, we obtain

$$\begin{aligned}\Delta_i^{(j)} &= S^{(2m-j-1)}(x_i + 0)\, S^{(j)}(x_i + 0) - S^{(2m-j-1)}(x_i - 0)\, S^{(j)}(x_i - 0) \\ &= S^{(j)}(x_i)\, \{S^{(2m-j-1)}(x_i + 0) - S^{(2m-j-1)}(x_i - 0)\}.\end{aligned}$$

This last expression for $\Delta_i^{(j)}$ vanishes for the following reasons:

(i) If $\epsilon_{ij}^* = 1$ then $S^{(j)}(x_i) = 0$ by (4.2) because $(i, j) \in e$.

(ii) If $\epsilon_{ij}^* = 0$ then by condition IV $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ and the difference inside the curly brackets vanishes.

3. The finite parts at x_0 and x_{k+1} also vanish in view of condition II. Thus (4.4) is established. (4.4) shows that

$$S^{(m)}(x) = 0 \qquad \text{for all real } x.$$

Therefore

$$S(x) \in \pi_{m-1}\,. \tag{4.5}$$

However, by assumption the problem (1.2) is m-poised. Now (4.5) and the set of equations (4.2) directly imply the desired conclusion (4.3). The existence and uniqueness of a g-spline solution $S(x) \in \mathscr{S}_m$ of an m-poised HB-problem (1.2) is hereby established.

Corollary 1. *If the HB-problem (1.2) is normal or n-poised we may choose $m = n$, when*

$$\mathscr{S}_n(E^*, x_1, ..., x_k) = \pi_{n-1}\,. \tag{4.6}$$

Proof. This follows readily from Theorem 1. In the first place we have the inclusion

$$\pi_{n-1} \subset \mathscr{S}_n\,.$$

If $S_0(x) \in \mathscr{S}_n$ then the HB-problem

$$f^{(j)}(x_i) = S_0^{(j)}(x_i), \qquad (i, j) \in e,$$

has the evident solution $f(x) = S_0(x)$. However, being normal it also has a polynomial solution $P(x) \in \pi_{n-1}$. From unicity we conclude that $S_0(x) = P(x)$ and (4.6) is established.

5. THE g-SPLINE INTERPOLATION FORMULA. It is convenient to summarize our results as follows: Under the assumptions of Theorem 1 we now define within $\mathscr{S}_m$ the g-splines $L_{ij}(x)$ satisfying the relations (1.8). If $f(x) \in C^\ell$ we may write

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i) L_{ij}(x) + Rf, \tag{5.1}$$

where the right-hand sum represents the g-spline interpolating $f(x)$ at the data of our HB-problem (1.2). We refer to (5.1) as the *g-spline interpolation formula.* This formula is exact for all elements of $\mathscr{S}_m$ and in particular for the elements of π_{m-1}.

6. THE OPTIMAL PROPERTY OF INTERPOLATING g-SPLINES.

THEOREM 2. *Let $I = [x_0, x_{k+1}]$ such that $x_0 < x_1$, $x_k < x_{k+1}$ and let $f(x) \in C^m(I)$, with $f^{(m-1)}(x)$ absolutely continuous and $f^{(m)}(x) \in L_2(I)$. We assume the Hermite-Birkhoff problem (1.2) to be m-poised, $\ell < m \leqslant n$, and let $S(x)$ be the unique g-spline satisfying the equations*

$$S^{(j)}(x_i) = f^{(j)}(x_i), \qquad (i,j) \in e. \tag{6.1}$$

Then

$$\int_I (f^{(m)}(x))^2\, dx > \int_I (S^{(m)}(x))^2\, dx,$$

unless $f(x) = S(x)$ in I.

PROOF. We shall derive our conclusion from the identity

$$\int_I (f^{(m)}(x))^2\, dx = \int_I (S^{(m)}(x))^2\, dx + \int_I (f^{(m)}(x) - S^{(m)}(x))^2\, dx, \tag{6.2}$$

which is an easy consequence of the relation

$$\int_I S^{(m)}(x)\, (S^{(m)}(x) - f^{(m)}(x)\, dx = 0. \tag{6.3}$$

This last relation is now established in precisely the same way as the relation (4.4) of Section 4. The only difference is that the second factor $S^{(m)}$ in

$$J = \int_I S^{(m)} S^{(m)}\, dx$$

is now to be replaced by the mth derivative of

$$s(x) = S(x) - f(x).$$

However, also $s(x)$ satisfies the HB "zero system"

$$s^{(j)}(x_i) = 0 \qquad (i, j) \in e \tag{6.4}$$

and the proof goes as before for precisely the same reasons all along.

If equality holds in (6.2) then $s(x) = S(x) - f(x) \in \pi_{m-1}$. Now (6.4) implies that $s(x) \equiv 0$ because our problem is m-poised.

II. The Best Approximations of Linear Functionals

7. The Newton-Cotes Procedure of Best Approximation. Let $I = [a, b]$ be a finite interval containing the reals (1.1) and let us consider a linear functional

$$\mathscr{L}f : C^{\ell}[a, b] \to R$$

of the form

$$\mathscr{L}f = \sum_{j=0}^{\ell} \int_a^b a_j(x) f^{(j)}(x)\, dx + \sum_{j=0}^{\ell} \sum_{i=1}^{n_j} b_{ji} f^{(j)}(x_{ji}), \tag{7.1}$$

where the $a_j(x)$ are piecewise continuous in I, $x_{ji} \in I$ and b_{ji} are real constants. (1.2) being an Hermite-Birkhoff interpolation problem, we propose to determine the reals B_{ij} such that the right-hand sum in the formula

$$\mathscr{L}f = \sum_{(i,j)\in e} B_{ij} f^{(j)}(x_i) + Rf \tag{7.2}$$

should represent an approximation to $\mathscr{L}f$ which is *best* in some sense.

A procedure which may be associated with the names of Newton and Cotes is as follows: There are $n = \sum \epsilon_{ij}$ parameters B_{ij}. We may therefore require that the formula (7.2) should be exact (i.e., $Rf = 0$) if $f \in \pi_{n-1}$. If B_{ij} are the constants determined by this condition, then (7.2) represents the best approximation of $\mathscr{L}f$ in the Newton-Cotes sense. This is the classical procedure of elementary numerical analysis.

In order to derive this approximation we substitute

$$f(x) = \frac{x^\nu}{\nu!} \qquad (\nu = 0, \ldots, n - 1)$$

into (7.2) (with $Rf = 0$) and obtain for the determination of the B_{ij} the following system of n equations in n unknowns

$$\mathscr{L} \frac{x^\nu}{\nu!} = \sum_{(i,j)\in e} B_{ij} \frac{x_i^{\nu-j}}{(\nu - j)!} \qquad (\nu = 0, \ldots, n - 1) \tag{7.3}$$

A glance at the matrix (2.8) shows that (7.3) is nonsingular if and only if the HB-problem (1.2) is *normal*, or *n-poised.*

If (1.2) is n-poised, we can alternatively procede as follows: We turn to the Hermite-Birkhoff interpolation formula (1.9) and observe that

$$f(x) = \sum_{(i,j)\in e} f^{(j)}(x_i) L_{ij}(x) \qquad \text{if} \qquad f \in \pi_{n-1} . \tag{7.4}$$

Operating on both sides with $\mathscr{L}$ we obtain the identity

$$\mathscr{L}f = \sum_{(i,j)\in e} f^{(j)}(x_i)\, \mathscr{L}L_{ij}(x) \qquad \text{if} \qquad f \in \pi_{n-1} , \tag{7.5}$$

showing that the formulas

$$B_{ij} = \mathscr{L}L_{ij}(x) \qquad (i,j) \in e \tag{7.6}$$

produce the coefficients in the Newton-Cotes best approximation of $\mathscr{L}f$.

8. Sard's Procedure of Best Approximation. Whether (1.2) is n-poised or not, let us assume that m, satisfying

$$\ell < m < n, \tag{8.1}$$

is such that

$$\textit{the HB-problem (1.2) is m-poised.} \tag{8.2}$$

We now require the approximation (7.2) to be exact if $f \in \pi_{m-1}$. This requirement is equivalent to the system of equations

$$\mathscr{L}\frac{x^\nu}{\nu!} = \sum_{(i,j)\in e} B_{ij} \frac{x_i^{\nu-j}}{(\nu-j)!} \qquad (\nu = 0,\ldots, m-1) \tag{8.3}$$

and (2.8) shows that the *matrix of this system has rank m.* Assuming (8.3) to hold we still have $n - m$ free parameters among the B_{ij} . However, Rf, defined by (7.2), is also a linear functional of the form (7.1) with the property that

$$Rf = 0 \qquad \text{if} \qquad f \in \pi_{m-1} . \tag{8.4}$$

Assuming $f(x) \in C^m[a, b]$ we may therefore write by Peano's theorem

$$Rf = \int_I K(x) f^{(m)}(x)\, dx, \tag{8.5}$$

with a kernel $K(x)$ which depends on the $n - m$ free parameters among the

B_{ij} but not on $f(x)$. These $n - m$ parameters are now determined by the requirement that

$$\int_I (K(x))^2 \, dx = \text{minimum}. \tag{8.6}$$

The B_{ij}, thereby completely determined, are substituted into (7.2) thereby producing the best approximation of $\mathscr{L}f$, of order m, in the sense of Sard [3].

9. Best Approximations and g-Splines. The main result of Part II is the following:

Theorem 3. *If the assumptions (8.1) and (8.2) hold, then Sard's best approximation (7.2) to $\mathscr{L}f$, of order m, is obtained by operating with $\mathscr{L}$ on both sides of the g-spline interpolation formula (5.1) of order m. In other words, the coefficients B_{ij} obtained as solutions of the minimum problem (8.6) with the m side conditions (8.3) are*

$$B_{ij} = \mathscr{L}L_{ij}(x), \tag{9.1}$$

where the $L_{ij}(x)$ are the fundamental functions of (5.1).

Proof. We wish to compare the functional

$$Rf = \mathscr{L}f - \sum_e B_{ij} f^{(j)}(x_i), \tag{9.2}$$

where the B_{ij} are defined by (9.1), with the functional

$$\tilde{R}f = \mathscr{L}f - \sum_e \tilde{B}_{ij} f^{(j)}(x_i) \tag{9.3}$$

whose coefficients $\tilde{B}_{ij}$ are only required to satisfy the m Eqs. (8.3). Evidently

$$Rf = 0 \quad \text{and} \quad \tilde{R}f = 0 \quad \text{if} \quad f \in \pi_{m-1}. \tag{9.4}$$

By Peano's theorem, if $f^{(m)}(x)$ exists and is integrable, we obtain the representations

$$Rf = \int_I K(x) f^{(m)}(x) \, dx, \qquad \tilde{R}f = \int_I \tilde{K}(x) f^{(m)}(x) \, dx \tag{9.5}$$

where the formulas

$$K(x) = R_t \frac{(t-x)_+^{m-1}}{(m-1)!}, \qquad \tilde{K}(x) = \tilde{R}_t \frac{(t-x)_+^{m-1}}{(m-1)!} \tag{9.6}$$

define these kernels for all real x.

Let us consider their difference

$$\sigma(x) = \tilde{K}(x) - K(x)$$

and observe that by (9.6), (9.2), and (9.3) it may be written as

$$\sigma(x) = (R_t - \tilde{R}_t) \frac{(t-x)_+^{m-1}}{(m-1)!} = \sum_e c_{ij} \frac{\partial^j}{\partial t^j} \frac{(t-x)_+^{m-1}}{(m-1)!} \bigg|_{t=x_i}, \qquad (9.7)$$

where

$$c_{ij} = B_{ij} - \tilde{B}_{ij} . \qquad (9.8)$$

We know by (9.6), in view of (9.4), that $K(x)$ and $\tilde{K}(x)$ vanish outside $I = [a, b]$, for $x > b$ because there $(t-x)_+^{m-1} = 0$ $(t \in I)$ and for $x < a$ because then $(t-x)_+^{m-1}$ is in $a \leqslant t \leqslant b$ a polynomial in t of degree $m-1$. $K(x)$ and $\tilde{K}(x)$ need not vanish in $[a, x_1]$, or in $[x_k, b]$. However, their difference $\sigma(x)$ vanishes also in these two intervals. This is evident if $x \in [x_k, b]$ from the last expression (9.7) for $\sigma(x)$ and becomes equally clear for $x \in [a, x_1]$ if we use instead of (9.6) the equivalent expressions

$$K(x) = (-1)^m R_t \frac{(x-t)_+^{m-1}}{(m-1)!}, \qquad \tilde{K}(x) = (-1)^m \tilde{R}_t \frac{(x-t)_+^{m-1}}{(m-1)!} .$$

Thus

$$\sigma(x) = 0 \qquad \text{everywhere outside} \qquad [x_1, x_k]. \qquad (9.9)$$

Let us now consider a function $S(x)$ satisfying

$$S^{(m)}(x) = \tilde{K}(x) - K(x). \qquad (9.10)$$

It is clear from (9.7) and (9.9) that $S(x)$ satisfies the conditions I and II in the Definition 3 of the class $\mathscr{S}_m$ of g-splines. Evidently the condition III, that is $S(x) \in C^{m-1}(-\infty, \infty)$, is also satisfied. Let us finally verify condition IV and thus conclude that

$$S(x) \in \mathscr{S}_m . \qquad (9.11)$$

For this purpose we look at those terms of the sum (9.7) which correspond to the same node x_i (i fixed); they are

$$\sum c_{i,j'} \frac{(x_i - x)_+^{m-j'-1}}{(m-j'-1)!}, \qquad (i, j') \in e.$$

Its only *discontinuous* derivatives are D^{m-j-1}, where j is such that $(i, j) \in e$. However

$$S^{(2m-j-1)}(x) = \sigma^{(m-j-1)}(x)$$

and this is therefore continuous at x_i if $(i,j) \notin e$ or $\epsilon_{ij}^* = 0$. This is precisely condition IV and (9.11) is established.

On the other hand, we know that the g-spline interpolation formula (5.1) is exact for all $f(x) \in \mathscr{S}_m$. But then also the approximation formula

$$\mathscr{L}f = \sum_e B_{ij} f^{(j)}(x_i) + Rf$$

obtained from (5.1) by operating on both sides with $\mathscr{L}$, must also be exact if $f(x) \in \mathscr{S}_m$. If we therefore substitute our g-spline $f(x) = S(x)$ into the identity

$$f(x) = \sum_{(i,j)\in e} B_{ij} f^{(j)}(x_i) + \int_I K(x) f^{(m)}(x)\, dx,$$

we know that the remainder term must vanish. By (9.10) we therefore conclude that

$$\int_I K(x)\,(\tilde{K}(x) - K(x))\, dx = 0$$

and a direct consequence of this is the relation

$$\int_I (\tilde{K}(x))^2\, dx = \int_I (K(x))^2\, dx + \int_I (\tilde{K}(x) - K(x_1)^2\, dx. \qquad (9.12)$$

From this we obtain the desired inequality

$$\int_I (\tilde{K}(x))^2\, dx > \int_I (K(x))^2\, dx$$

unless, by (9.12), $\sigma(x)$ vanishes identically, whence $c_{ij} = 0$ and by (9.8), that the two approximations having coefficients $\tilde{B}_{ij}$ and B_{ij}, respectively, are identical. This concludes our proof of Theorem 3. The method of the proof just given was first used by Greville in establishing Theorem 3 for the simplest case of Lagrange data [6].

As in the case of Lagrange data [2] we now obtain for the "interpolation functional" $\mathscr{L}f = f(x)$ the following

COROLLARY 2. *The g-spline interpolation formula (5.1) is also the best interpolation of order m in the sense of Sard for the Hermite-Birkhoff problem (1.2).*

III. Examples, Special Cases, Applications and Unsolved Problems

10. On the Construction of the g-Spline Interpolation Formula. The procedure described in the first paragraph of the proof of Theorem 1 is not well suited for the actual construction of the formula (5.1) in any specific case. A more efficient way leading to a system of only $m + n$ equations (instead of $2mk$) is as follows.

From conditions I, II, III of Definition 3 it is clear that a g-spline $S(x)$ must be of the form

$$S(x) = P_{m-1}(x) + \sum_{i=1}^{k} \sum_{j=0}^{m-1} c_{ij} \frac{(x - x_i)_+^{2m-j-1}}{(2m - j - 1)!} \tag{10.1}$$

where $P_{m-1}(x) \in \pi_{m-1}$ while the c_{ij} are constants. Conversely, any function (10.1) satisfies these conditions with the exception of the requirement that

$$S(x) \in \pi_{m-1} \qquad \text{if} \qquad x_k < x. \tag{10.2}$$

Let us enforce also condition IV: From (10.1) we see that $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$ if and only if $c_{ij} = 0$, while condition IV requires that this be the case if and only if $\epsilon_{ij}^* = 0$. Leaving out all such terms we obtain

$$S(x) = P_{m-1}(x) + \sum_{(i,j)\in e} c_{ij} \frac{(x - x_i)_+^{2m-j-1}}{(2m - j - 1)!} \tag{10.3}$$

as the appropriate expression. To insure also (10.2) we expand all binomials and equating to zero the coefficients of $x^m, x^{m+1},\ldots, x^{2m-1}$, we obtain the equations

$$\sum_{\substack{(i,j)\in e \\ j\leqslant \nu}} \frac{c_{ij}}{(2m - j - 1)!} \binom{2m - j - 1}{2m - \nu - 1} (- x_i)^{\nu-j} = 0 \qquad (\nu = 0,\ldots, m - 1). \tag{10.4}$$

If we consider also the n equations of the m-poised HB-problem

$$S^{(j)}(x_i) = y_i^{(j)}, \qquad (i, j) \in e, \tag{10.5}$$

we obtain $m + n$ Eqs. (10.4), (10.5), which if solved will produce the unique interpolating g-spline $S(x)$. Writing the solution so as to exhibit the $f^{(j)}(x_i)$, we obtain the interpolating g-spline

$$S(x) = \sum_{(i,j)\in e} y_i^{(j)} L_{ij}(x).$$

EXAMPLE. We know from (2.7) that the HB-problem

$$f(-1) = y_1, \quad f'(0) = y_2', \quad f(1) = y_3 \tag{10.6}$$

is 2-poised. Thus $\ell = 1$, $m = 2$, $n = 3$. The above procedure will show the interpolating g-spline to be

$$S(x) = y_1 L_{10}(x) + y_2' L_{21}(x) + y_3 L_{30}(x), \tag{10.7}$$

where

$$\begin{aligned} L_{10}(x) &= \tfrac{1}{4}(1 - 3x) + \tfrac{1}{4}(x+1)_+^3 - \tfrac{3}{2}x_+^2 - \tfrac{1}{4}(x-1)_+^3, \\ L_{21}(x) &= -\tfrac{1}{2}(1 + x) + \tfrac{1}{2}(x+1)_+^3 - 3x_+^2 - \tfrac{1}{2}(x-1)_+^3, \\ L_{30}(x) &= \tfrac{3}{4}(1 + x) - \tfrac{1}{4}(x+1)_+^3 + \tfrac{3}{2}x_+^2 + \tfrac{1}{4}(x-1)_+^3. \end{aligned} \tag{10.8}$$

By direct evaluations we find from (10.7) that

$$\int_{-1}^{1} (S''(x))^2 \, dx = 6\left(y_2' - \frac{y_3 - y_1}{2}\right)^2,$$

$$\int_0^1 f(x)\, dx = \frac{1}{16}(5y_1 + 2y_2' + 11y_3).$$

Applying Theorems 2 and 3 to the HB-problem (10.6), for $m = 2$, we obtain the following

COROLLARY 3. (i) *If $f''(x) \in L_2(-1, 1)$ then*

$$\int_{-1}^{1} (f''(x))^2 \, dx \geqslant 6\left(f'(0) - \frac{f(1) - f(-1)}{2}\right)^2$$

with equality if and only if $F(x)$ is of the form (10.7).

(ii) *The quadrature formula*

$$\int_0^1 f(x)\, dx = \frac{1}{16}(5f(-1) + 2f'(0) + 11f(1)) + Rf$$

is the best among all formulas of this type which are exact for linear functions.

11. THE CASE OF QUASI-HERMITE PROBLEMS.

DEFINITION 4. *We say that the Hermite-Birkhoff problem (1.2) is a* *quasi-Hermite problem, provided that either $k = 2$ or if $k > 2$ then the partial problem*

$$f^{(j)}(x_i) = y_i^{(j)}, \qquad (i, j) \in e, \qquad (i = 2, 3,\dots, k-1), \tag{11.1}$$

obtained from (1.2) by omitting the data pertaining to the extreme nodes x_1 and x_k, is an Hermite problem. We will refer to such problems as qH-problems.

Such problems occur frequently. For instance the data appearing on the right side of a finite Euler summation formula, such as (3), are the data of a qH-system.

Following Polya [11] we introduce in connection with (1.2) the quantities

$$M_\nu = \sum_{j \leqslant \nu} \epsilon_{ij} \qquad (\nu = 0, 1,\dots, \ell), \tag{11.2}$$

which count the number of those equations (1.2) involving derivatives of order not exceeding ν. Evidently $M_\ell = n$.

G. Polya established in [11] the following

THEOREM OF POLYA. *If $k = 2$ then the HB-problem (1.2) is normal, or n-poised, if and only if the inequalities*

$$M_0 \geqslant 1, \qquad M_1 \geqslant 2,\dots, M_{\nu-1} \geqslant \nu,\dots, M_{\ell-1} \geqslant \ell \tag{11.3}$$

hold.

In [12] I have extended Polya's result as follows

THEOREM 4. *A quasi-Hermite problem (1.2) is normal, or n-poised, if and only if Polya's inequalities (11.3) hold.*

The proof as given in [12] actually yields more and we may state

THEOREM 5. *If (1.2) is quasi-Hermite and $\ell < m \leqslant n$ (see (1.7)), then (1.2) is m-poised if and only if Polya's inequalities (11.3) hold. In other words: If the qH-problem (1.2) is m-poised for a single value of m such that*

$$\ell < m \leqslant n, \tag{11.4}$$

then it is n-poised and therefore also m-poised for all values of m satisfying (11.4).

There is nothing to change in our proof of Theorem 4 as given in [12] to yield Theorem 5 beyond replacing the integer n appearing in [12, Section 2] by the integer m satisfying (11.4).

Theorem 5 makes it easy to check whether a qH-problem is m-poised or not. We shall have frequent occasions to use it, as all our further examples will be quasi-Hermite interpolation problems.

12. LIDSTONE'S SERIES. G. J. Lidstone [13] found the following formal expansion

$$f(x) = \sum_0^\infty f^{(2\nu)}(1)\, \Lambda_\nu(x) + \sum_0^\infty f^{(2\nu)}(0)\, \Lambda_\nu(1-x), \tag{12.1}$$

where $\Lambda_\nu(x)$ are odd polynomials of degree $2\nu + 1$, defined by the generating function

$$\frac{\sinh xt}{\sinh t} = \sum_0^\infty t^{2\nu} \Lambda_\nu(x). \tag{12.2}$$

In the monograph of Boas and Buck [14, p. 14] it is shown as part of a comprehensive theory that the expansion (12.1) is valid for any entire function of exponential type $< \pi$.

Let us examine the quasi-Hermite problem

$$f^{(2\nu)}(0) = y_1^{(2\nu)}, \qquad f^{(2\nu)}(1) = y_2^{(2\nu)}, \qquad (\nu = 0, 1, \ldots, h), \tag{12.3}$$

corresponding to the incidence matrix

$$E = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{Vmatrix}. \tag{12.4}$$

Here $k = 2$, $\ell = 2h$, $n = 2h + 2$. Polya's inequalities (11.3) are evidently satisfied. The problem (12.3) is therefore n-poised and its unique solution $f(x) \in \pi_{2h+1}$ is evidently given by the section

$$f(x) = \sum_0^h y_2^{(2\nu)} \Lambda_\nu(x) + \sum_0^h y_1^{(2\nu)} \Lambda_\nu(1-x) \tag{12.5}$$

of Lidstone's series (12.1).

By Theorem 5, in particular (11.4), we may also choose the value

$$m = 2h + 1.$$

Again (12.3) is m-poised so that we may apply Theorem 1 and obtain as solution of (12.3) an element $S(x)$ of the class $\mathscr{S}_m$ of g-splines. In this case $E = E^*$ and (12.4) shows that

$$\epsilon_{ij} = \epsilon_{ij}^* = 0 \qquad \text{whenever } j \text{ is odd} = 1, 3, \ldots, 2h - 1.$$

By Definition 3 we concluded that

$$S^{(2m-j-1)}(x) \text{ is continuous at } x = 0 \text{ and } x = 1 \text{ if}$$

$$j = 1, 3, \ldots, 2h - 1,$$

or, equivalently, that

$$S^{(\nu)}(x) \text{ is continuous at } x = 0 \text{ and } x = 1 \text{ if } \nu = 2h + 2, 2h + 4, \ldots, 4h. \tag{12.6}$$

On the other hand (condition II of Definition 3)

$$S(x) \in \pi_{m-1} = \pi_{2h} \text{ in } (-\infty, 0) \text{ and also in } (1, +\infty). \tag{12.7}$$

We conclude from (12.6) and (12.7) that

$$S^{(\nu)}(+0) = S^{(\nu)}(1-0) = 0 \quad \text{if} \quad \nu = 2h + 2, 2h + 4, \ldots, 4h - 2, 4h. \tag{12.8}$$

Moreover (condition I)

$$S(x) \in \pi_{2m-1} = \pi_{4h+1} \quad \text{if} \quad 0 < x < 1. \tag{12.9}$$

We now restrict $S(x)$ to $(0, 1)$ and wish to prove that

$$S(x) \in \pi_{2h+1} \quad \textit{if} \quad 0 < x < 1. \tag{12.10}$$

Proof. Indeed, $S^{(4h)}(x) \in \pi_1$ by (12.9), while $S^{(4h)}(x)$ vanishes for $x = 0$ and $x = 1$ by (12.8). This clearly implies that $S^{(4h)}(x) \equiv 0$ and therefore that

$$S(x) \in \pi_{4h-1}\,.$$

Again (12.8) for $\nu = 4h - 2$ shows that

$$S(x) \in \pi_{4h-3}\,.$$

In this way (by complete induction if necessary!) we conclude that (12.10) indeed holds.

Now it is clear, by (12.10), that the g-spline solution $S(x)$ of (12.3), for $m = 2h + 1$, coincides in $(0, 1)$ with the section (12.5) of Lidstone's series. This identification transfers the minimal properties of the g-spline $S(x)$ to the section of Lidstone's series. Applying Theorem 2 with $I = [0, 1]$ we obtain

Theorem 6. *The section (12.5) of Lidstone's series minimizes the integral*

$$\int_0^1 (f^{(2h+1)}(x))^2\, dx$$

among all solutions of the Hermite-Birkhoff problem (12.3) such that $f^{(2h+1)}(x) \in L_2(0, 1)$.

As a matter of fact we can also apply our Theorem 3. For the interpolation functional $\mathscr{L}f = f(x)$ we obtain

THEOREM 6'. *Lidstone's polynomial (12.5) gives for every fixed x of the interval $0 \leqslant x \leqslant 1$ the best solution of the interpolation problem (12.3) among all expression of the form*

$$f(x) = \sum_0^h y_2^{(2\nu)} A_\nu(x) + \sum_0^h y_1^{(2\nu)} B_\nu(x)$$

which reproduce exactly all solution of (12.3) which are in π_{2h}.

If x is outside $[0, 1]$ this is no longer true since the g-spline $S(x)$ then assumes its rightful role of the best interpolation formula (Corollary 2).

Let us finally apply Theorem 3 to the functional

$$\mathscr{L}f = \int_0^1 f(x)\, dx.$$

Integrating (12.2) with respect to x from $x = 0$ to $x = 1$ we find that the integrals

$$C_\nu = \int_0^1 A_\nu(x)\, dx = \int_0^1 A_\nu(1 - x)\, dx, \qquad (\nu = 0, 1, \ldots),$$

are the expansion coefficients in

$$\tanh \frac{t}{2} = \sum_0^\infty C_\nu t^{2\nu+1}.$$

Integrating the section (12.5) and applying Theorem 3 we obtain

THEOREM 6''.

$$\int_0^1 f(x)\, dx = \sum_0^h C_\nu (f^{(2\nu)}(0) + f^{(2\nu)}(1)) + Rf \tag{12.11}$$

is the best quadrature formula among all formulas of this type which are exact for all $f(x) \in \pi_{2h}$.

Observe that there is a 1-parameter family of formulae of the type (12.11) exact in π_{2h}.

13. AN ANALOGUE OF LIDSTONE'S EXPANSION IF $k > 2$. The expansion (12.1) is a 2-point expansion. Let us now take $k = 3$ and consider the infinite interpolation problem

$$f(0) = y_2, \qquad f^{(2\nu)}(-1) = y_1^{(2\nu)}, \qquad f^{(2\nu)}(1) = y_3^{(2\nu)} \qquad (\nu = 0, 1, \ldots). \tag{13.1}$$

If we truncate this system so as to include the derivatives of order $\ell = 2h$ we obtain the quasi-Hermite problem

$$f(0) = y_2, \qquad f^{(2\nu)}(-1) = y_1^{(2\nu)}, \qquad f^{(2\nu)}(1) = y_3^{(2\nu)}, \qquad (\nu = 0, 1, \ldots, h) \tag{13.2}$$

corresponding to the matrix

$$E = \begin{Vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \end{Vmatrix}. \tag{13.3}$$

Here $k = 3$, $\ell = 2h$, $n = \sum \epsilon_{ij} = 2h + 3$. The inequalities (11.3) being visibly satisfied we may apply Theorem 5 and conclude that (13.2) is m-poised for each of the three values

$$m = 2h + 1, \qquad m = 2h + 2, \qquad m = 2h + 3. \tag{13.4}$$

The first two values (13.4) lead to g-spline interpolation formulas, while the last, where $m = n$, produces the polynomial Hermite-Birkhoff interpolation formula of degree $n - 1 = 2h + 2$.

For the problem (13.1) we no longer have a basic series of polynomials in the sense of J. M. Whittaker [15]. Indeed, it is easily seen that there is no polynomial $P(x)$ which will satisfy (13.1) with all right-hand sides $= 0$, except $y_2 = 1$. This also follows from a general result [15, Theorem 33, p. 46] which is related to the results of [11] and [12].

Nevertheless there is now an expansion formula

$$f(x) = f(0)\, A_0(x) + \sum_0^\infty f^{(2\nu)}(1)\, B_\nu(x) + \sum_0^\infty f^{(2\nu)}(-1)\, B_\nu(-x), \tag{13.5}$$

where $A_0(x)$ and $B_\nu(x)$ are entire functions of exponential type $\pi/2$. Clearly

$$A_0(x) = \cos \frac{\pi x}{2}. \tag{13.6}$$

Setting $F(x) = e^{xt}$ in (13.5) we readily find the generating function

$$\frac{\sinh t(x+1)}{\sinh 2t} - \frac{1}{2 \cosh t} \cos \frac{\pi x}{2} = \sum_0^\infty t^{2\nu} B_\nu(x). \tag{13.7}$$

Using (12.2) and the expansion

$$\frac{1}{\cosh t} = \sum_0^\infty D_\nu t^{2\nu},$$

we find that

$$B_\nu(x) = 2^{2\nu} A_\nu \left(\frac{1+x}{2}\right) - \frac{1}{2} D_\nu \cos \frac{\pi x}{2} \tag{13.8}$$

Thus

$$B_0(x) = \frac{1+x}{2} - \frac{1}{2} \cos \frac{\pi x}{2}.$$

It is easily verified from (13.6) and (13.8) that the expansion (13.5) is valid for any *polynomial* $f(x)$.

Here are a few open questions:

1. *Does (13.5) hold for all entire functions $f(x)$ of exponential type $<\pi$?*

That the bound π, if correct, can not be increased is shown by $f(x) = \sin \pi x$ for which the right side of (13.5) vanishes identically.

2. *Do the three interpolation formulas corresponding to the values (13.4) converge (termwise) to the expansion (13.5) as $h \to \infty$?*

Similar questions arise for an obvious extension of (13.1) to an arbitrary finite value of k. For attempts to answer these questions the minimal property of Theorem 2 may well play an essential role.

14. Euler Interpolation Problems. Let us finally consider the k-point interpolation problem corresponding to the matrix

$$E = \left\| \begin{matrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \end{matrix} \right\| \quad (k \text{ rows, } 2h \text{ columns})$$

and the nodes $0, 1, \ldots, k-1 = r$. Here $\ell = 2h - 1$, $n = 2h + k$. If we choose $m = 2h$, which is evidently admissible, we obtain as solution a g-spline interpolation formula which by integration *must* yield Euler's summation formula, in view of our Theorem 3 and [1, Theorem 2]. By Theorem 5 the further admissible values of m are

$$m = 2h + 1, 2h + 2, \ldots, 2h + k = n.$$

3. *Do all these formulas converge termwise, as $h \to \infty$, to an infinite expansion in terms of functions of exponential type analogous to (13.5)?*

A generating function for the coefficients of this expansion is easily derived.

We conclude by raising a final question: Throughout this paper we have always selected a value m satisfying (11.4) and such that (1.2) is m-poised.

4. *What are the precise results which are to replace Theorem 1 and 2 if (11.4) holds, while (1.2) is not m-poised?*

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